1.7 Linear Independence

This concept can be well illustrated in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Let's Consider

$\boxed{\text{CASE 1}}$ $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

and the equation

$$x_1 \vec{u} + x_2 \vec{v} = \vec{0}$$

the solution for this vector equation is obtained as usual

$$x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ which is equivalent to

Augmented matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 0 \end{pmatrix}$$

or $-3x_2 = 0 \Rightarrow x_2 = 0$, $\quad \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ unique soln.

$x_1 + 2\overset{=0}{x_2} = 0 \Rightarrow x_1 = 0$,

Consider now, the vectors

$\boxed{\text{CASE 2}}$ $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and the equation $x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \vec{0}$

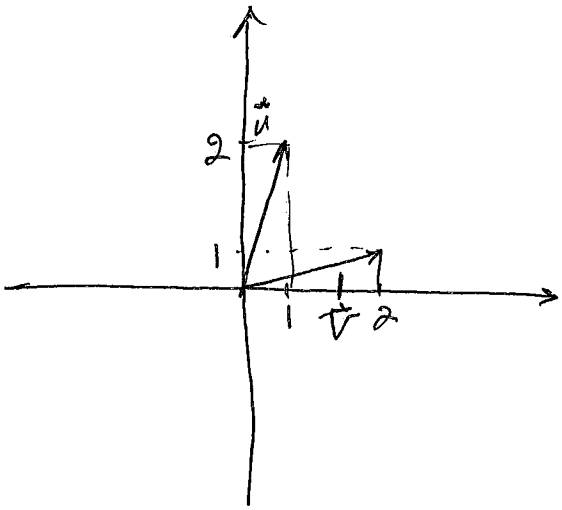
Augmented matrix

or $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

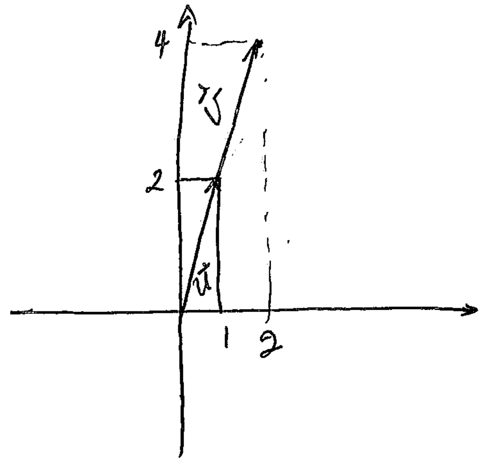
or $x_1 = -2\underbrace{x_2}_{\text{free Variable}}$ infinitely many solutions.

Geometric Interpretation

CASE 1



CASE 2



$\vec{u}$ and $\vec{v}$ not in the same line in $\mathbb{R}^2$, they are said to be "linearly independent"

$\vec{u}$ and $\vec{v}$ in the same line in $\mathbb{R}^2$, they are said to be "linearly dependent".

From Case 1, we conclude that

$$x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ if and only if } x_1 = 0,\ x_2 = 0$$

However, Case 2 shows that there are $x_1 \neq 0$ or $x_2 \neq 0$ such that

$$x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For example, $\vec{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, or $\vec{x} = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}$

Since $-2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (1) \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

General Concepts

Def. A set of vectors $\{\vec{v}_1, \vec{v}_2, .., \vec{v}_p\}$ is said to be

a) linearly independent if the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdot\cdot + x_p\vec{v}_p = \vec{0}$$

has only the trivial solution

b) A set of vectors $\{\vec{v}_1, \vec{v}_2, .., \vec{v}_p\}$ is said to be linearly dependent if there exists weights $c_1, c_2, .., c_p$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdot\cdot + c_p\vec{v}_p = \vec{0}.$$

According to this definition and the previous results, it can be easily proved that

If $\vec{u}, \vec{v}$ in $\mathbb{R}^2$ are not in the same line they are linearly independent. Otherwise, they are linearly depend[ent]

Observations:

- The zero vector $\vec{0}$ in $\mathbb{R}^n$ is linearly dependent, since

$$c_1\vec{0} = \vec{0}, \quad \text{for any } c_1 \neq 0.$$

- A set of only one vector, $\vec{u} \neq \vec{0}$ in $\mathbb{R}^n$ is lin. indep

Since $x_1\vec{u} = 0$ if and only if $x_1 = 0$.

What can we say in $\mathbb{R}^3$?

Consider three vectors $\vec{u}, \vec{v}, \vec{w}$, all differents than the $\vec{0}$ vector.

then, they are linearly independent if any two of them are not in the same line and if the third of them does not lie in the same plane.

Make Graphs

An inmediate result from the definition of linear independence is that

The columns of A are linearly independent
if and only if
$A\vec{x}=\vec{0}$ has only the trivial solution. Why?

Also,

Statement (P)

then, $\boxed{S=\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_p\} \text{ is linearly dependent}}$ if and only if $\boxed{\text{at least one of the vectors in } S \text{ is a linear combination of the others}}$

Statement (Q)

Proof: $P \Rightarrow Q$.

Assume S is linearly dependent, then there are c_1, c_p not all zeros such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = \vec{0}$$

Assuming $c_j \neq 0$ and $j \neq 1, p$

$$c_j\vec{v}_j = -c_1\vec{v}_1 + \dots - c_p\vec{v}_p \Rightarrow \boxed{\vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 \dots - \frac{c_p}{c_j}\vec{v}_p}$$

If $j=1$ $\quad \boxed{\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2 \dots - \frac{c_p}{c_1}\vec{v}_p}$

$Q \Rightarrow P$.

If $\vec{V}_j$ is linear comb. of the others, we have

and $j \neq 1, p$ $\quad \vec{V}_j = d_1\vec{V}_1 + d_2\vec{V}_2 + \cdots + d_p\vec{V}_p$

then
$$-\vec{V}_j + d_1\vec{V}_1 + \cdots + d_p\vec{V}_p = \vec{0}.$$
$$(-1)\vec{V}_j + d_1\vec{V}_1 + \cdots + d_p\vec{V}_p = \vec{0}$$

If $j=1$ $\quad -V_1 + d_2\vec{V}_2 + \cdots + d_p\vec{V}_p = \vec{0}$

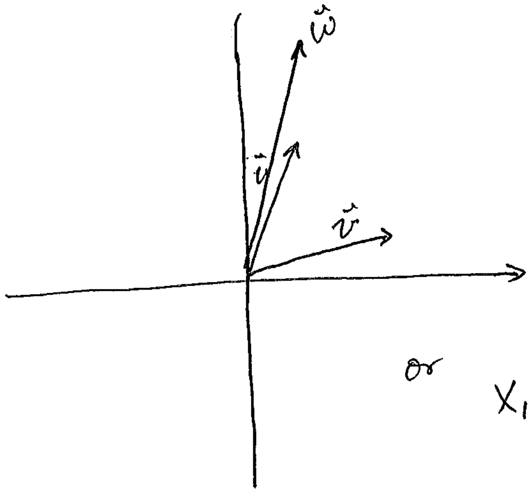
Similar if $j=p$ $\quad d_1\vec{V}_1 + d_2\vec{V}_2 + \cdots + d_{p-1}\vec{V}_{p-1} - \vec{V}_p = \vec{0}$

In these cases, not all the weights are zero (Since the weight of $\vec{V}_j$ is $c_j = -1$)

Therefore, the set $S = \{\vec{V}_1, \vec{V}_2, \ldots, \vec{V}_p\}$ is linearly dependent.

Consider the vectors $\vec{u} = \begin{pmatrix}1\\2\end{pmatrix}$, $\vec{v} = \begin{pmatrix}2\\1\end{pmatrix}$, $\vec{w} = \begin{pmatrix}1\\3\end{pmatrix}$ in $\mathbb{R}^2$

Is the set $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly dependent?



We need to find x_1, x_2, x_3 not all zeros such that

$$x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = \vec{0}$$

or $$x_1\begin{pmatrix}1\\2\end{pmatrix} + x_2\begin{pmatrix}2\\1\end{pmatrix} + x_3\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

or $$\begin{pmatrix}1 & 2 & 1\\2 & 1 & 3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + x_2 + 3x_3 = 0 \end{cases}$$

What we found from this representation is that there are more unknowns than equations. Thus, there must be a free variable. Therefore, the system has infinitely many solutions.

In fact, row-reducing the augmented matrix

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & -3 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5/3 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \end{pmatrix}$$

$$x_2 = \frac{1}{3}x_3 \qquad x_1 = -\frac{5}{3}x_3$$

So possible solutions are $\vec{x} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$

and $\vec{x} = \begin{pmatrix} -5/3 \\ 1/3 \\ 1 \end{pmatrix}$

Verification:

$$-5\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -5+5 \\ -10+1+9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Observation:

- the number of vectors in a linearly independent set in $\mathbb{R}^2$ is at most 2.

Thm 8

If a set has more vectors than there entries in each vector, then the set is linearly dependent. That is, any set $\{\vec{v}_1, .., \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p>n$.

Proof.

Consider $$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p \\ \vdots & & & \vdots \\ n & & & n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \vec{0}$$

$p>n$. therefore, there are more columns than rows which implies that there are free variables (at least $p-n$ free variables)

By choosing nonzero values for these free variables, we obtain a linear combination with nonzero weights

Therefore, the set $\{\vec{v}_1, \vec{v}_2, .., \vec{v}_p\}$ is linearly dependent.

Discuss example 6 in the book pag. 59.